STATISTICS ON ORDERED PARTITIONS OF SETS AND q-STIRLING NUMBERS

MASAO ISHIKAWA, ANISSE KASRAOUI, AND JIANG ZENG

Abstract. An ordered partition of $[n] := \{1, 2, \ldots, n\}$ is a sequence of its disjoint subsets whose union is $[n]$. The number of ordered partitions of $[n]$ with k blocks is $k!S(n,k)$, where $S(n,k)$ is the Stirling number of second kind. In this paper we prove some refinements of this formula by showing that the generating function of some statistics on the set of ordered partitions of $[n]$ with k blocks is a natural q-analogue of $k!S(n,k)$. In particular, we prove several conjectures of Steingrímsson. To this end, we construct a mapping from ordered partitions to walks in some digraphs and then, thanks to transfer-matrix method, we determine the corresponding generating functions by determinantal computations.

Contents

Keywords: ordered partitions, Euler-Mahonian statistics, q-Stirling numbers of second kind, transfert-matrix method.

MR Subject Classifications: Primary 05A18; Secondary 05A15, 05A30.

1. Introduction

An *(unordered) set partition* of $[n] = \{1, 2, \ldots, n\}$ is a collection of its disjoint subsets, called *blocks*, whose union is $[n]$. By convention, the standard notation of a partition of $[n]$ is $\pi_0 = B_1 - B_2 - \cdots - B_k$, where the blocks B_i are arranged in increasing order of their minimal elements and in each block B_i the elements are arranged in increasing order. Let $|\pi| = n$ if π is a partition of $[n]$. Let $\mathcal{P}_n^k$ be the set of partitions of $[n]$.

Version of 06/06/2006.

An *ordered partition* π of $[n]$ with k blocks is a rearrangement of blocks of a partition in $\mathcal{P}_n^k$. Namely $\pi = B_{\sigma(1)} - B_{\sigma(2)} - \cdots - B_{\sigma(k)}$, where σ is a permutation of $[k]$. We will say that σ is the permutation induced by π and set $\sigma = \operatorname{perm}(\pi)$.

Let $\mathcal{OP}_n^k$ be the set of ordered partitions of $[n]$ into k blocks, $\mathcal{OP}_n = \bigcup_{k\geq 1} \mathcal{OP}_n^k$ be the set of all ordered partitions of $[n]$, and $\mathcal{OP}^k = \bigcup_{n\geq 1} \mathcal{OP}_n^k$ be the set of all ordered partitions into k blocks. Clearly we have $|\mathcal{OP}_n| = \sum_{k=0}^{n} k!S(n,k)$, where $S(n,k)$ is the Stirling number of second kind and it is not hard to derive the following exponential generating function :

$$\sum_{n\geq 0} |\mathcal{OP}_n| \frac{z^n}{n!} = \frac{1}{2-e^z} = 1 + z + 3\,\frac{z^2}{3!} + 13\,\frac{z^3}{3!} + 75\,\frac{z^4}{4!} + \cdots .$$

Define the p,q-integer $[n]_{p,q} = \frac{p^n - q^n}{p-q}$, the p,q-factorial $[n]_{p,q}! = [1]_{p,q}[2]_{p,q}\cdots[n]_{p,q}$ and the p,q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!} \qquad n \geq k \geq 0.$$

If $p = 1$, we shall write $[n]_q$, $[n]_q!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ for $[n]_{1,q}$, $[n]_{1,q}!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_{1,q}$ respectively.

The following q-analogues of *Eulerian numbers* and *Stirling numbers of the second kind* were first introduced by Carlitz [1, 2].

The q-Eulerian numbers $\left\langle {n \atop k} \right\rangle_q$ $(n \geq k \geq 0)$ are defined by

$$\left\langle {n \atop k} \right\rangle_q = q^k [n-k]_q \left\langle {n-1 \atop k-1} \right\rangle_q + [k+1]_q \left\langle {n-1 \atop k} \right\rangle_q .$$

The first values of the q-Eulerian numbers $\left\langle {n \atop k} \right\rangle_q$ $(n \geq k \geq 0)$ read

$n \setminus k$	0	1	2	3
1	1			
2	1	q		
3	1	$2q + 2q^2$	q^3	
4	1	$3q + 5q^2 + 3q^3$	$3q^3 + 5q^4 + 3q^5$	q^3.

Let $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ be a permutation of $[n]$, the integer $i \in [n-1]$ is called a *descent* of σ if $\sigma(i) > \sigma(i+1)$. The *major index* of σ, noted $\operatorname{maj}\sigma$, is the sum of its descents, i.e., $\operatorname{maj}\sigma = \sum_i i$, where the summation is over all descents i of σ. Then Carlitz [2] gave the following combinatorial interpretation of q-Eulerian numbers:

$$\left\langle {n \atop k} \right\rangle_q = \sum_{\sigma} q^{\operatorname{maj}\sigma},$$

where the summation is over all permutations of $[n]$ with k descents.

The q-Stirling numbers $S_q(n,k)$ of the second kind are defined by:

$$S_q(n,\,k) = q^{k-1} S_q(n-1,\,k-1) + [k]_q S_q(n-1,\,k) \qquad (n \geq k \geq 0), \tag{1.1}$$

where $S_q(n,k) = \delta_{n\,k}$ if $n = 0$ or $k = 0$. The first values of the q-Stirling numbers $S_q(n,\, k)$ read

$n \setminus k$	1	2	3	4
1	1			
2	1	q		
3	1	$1+q+q^2$	q^3	
4	1	$1+3q+2q^2+q^3$	$q^2+2q^3+2q^4+q^5$	q^6.

There has been a considerable amount of recent interest in properties and combinatorial interpretations of the q-Eulerian numbers and q-Stirling numbers and related numbers (see e.g. [1, 2, 3, 6, 7, 8, 9, 10, 11, 13, 15, 14, 16]).

The following identity was derived in [17]:

$$[k]_q!\, S_q(n,k) = \sum_{m=1}^{k} q^{k(k-m)} \begin{bmatrix} n-m \\ n-k \end{bmatrix}_q \left\langle \begin{matrix} n \\ m-1 \end{matrix} \right\rangle_q . \tag{1.2}$$

In the aim to give a combinatorial proof of (1.2), Steingrímsson [13] introduced the following

Definition 1.1. *A statistic Stat on $\mathcal{OP}_n^k$ is called* Euler-Mahonian *if its generating function is equal to $[k]_q! S_q(n,k)$, i.e.,*

$$\sum_{\pi \in \mathcal{OP}_n^k} q^{Stat\,\pi} = [k]_q!\, S_q(n,k).$$

Steingrímsson [13] has found a few of *Euler-Mahonian* statistics and conjectured more such statistics on ordered partitions. From a different point view, Wachs [7] has also obtained some *Euler-Mahonian* statistics on ordered partitions. Although Zeng [18] has showed that much more such statistics can be derived from some classical bijections between ordered partitions and weighted Motzkin paths, it is not clear how to encode the conjectured statistics of Steingrímsson by the statistics obtained by this method.

It is the purpose of this paper to propose a new approach to attack such kind of problem. We shall construct a bijection ψ between ordered partitions and some walks in some digraphs (see section 3). This bijection keeps track of several statistics of Steingrímsson. Then, by transfer-matrix method, we evaluate the generating functions of these statistics on ordered partitions and prove that they are indeed Euler-Mahonian.

2. Definitions and main results

2.1. **Definitions.** Let $\pi = B_1 - B_2 - \cdots - B_k$ be a partition in $\mathcal{OP}_n^k$. The *opener* of a block in π is its least element and the *closer* is its greatest element. The sets of openers and closers of π are denoted by $\mathrm{open}(\pi)$ and $\mathrm{clos}(\pi)$, respectively. We define a *partial order* on blocks $B_i's$ as follows : $B_i > B_j$ if all the letters of B_i are greater than those of B_j; in other words, if the opener of B_i is greater than the closer of B_j. We say that i is a *block descent* in π if $B_i > B_{i+1}$. The *block major index* of π, denoted $\mathrm{bmaj}(\pi)$, is the sum of the block descents in π. A *block excedance* (resp. *block inversion*) in π is a pair (i,j) such that $i < j$ and $B_i < B_j$ (resp. $B_i > B_j$). We denote by $\mathrm{bExc}\pi$ (resp. $\mathrm{bInv}\pi$) the number of block excedances (resp. block inversions) in π. Let block(i) be the index of the block (counting from the left) containing i, namely the integer j such that $i \in B_j$.

Following Steingrímsson [13], for $1 \leq i \leq k$ we define ten coordinate statistics on $\pi \in \mathcal{OP}_n^k$:

$$\begin{aligned}
\mathrm{ros_i}(\pi) &= \#\{j \in \mathrm{open}(\pi) \,|\, \mathrm{i > j,\ block(j) > block(i)}\},\\
\mathrm{rob_i}(\pi) &= \#\{j \in \mathrm{open}(\pi) \,|\, \mathrm{i < j,\ block(j) > block(i)}\},\\
\mathrm{rcs_i}(\pi) &= \#\{j \in \mathrm{clos}(\pi) \,|\, \mathrm{i > j,\ block(j) > block(i)}\},\\
\mathrm{rcb_i}(\pi) &= \#\{j \in \mathrm{clos}(\pi) \,|\, \mathrm{i < j,\ block(j) > block(i)}\},\\
\mathrm{los_i}(\pi) &= \#\{j \in \mathrm{open}(\pi) \,|\, \mathrm{i > j,\ block(j) < block(i)}\},\\
\mathrm{lob_i}(\pi) &= \#\{j \in \mathrm{open}(\pi) \,|\, \mathrm{i < j,\ block(j) < block(i)}\},\\
\mathrm{lcs_i}(\pi) &= \#\{j \in \mathrm{clos}(\pi) \,|\, \mathrm{i > j,\ block(j) < block(i)}\},\\
\mathrm{lcb_i}(\pi) &= \#\{j \in \mathrm{clos}(\pi) \,|\, \mathrm{i < j,\ block(j) < block(i)}\},
\end{aligned}$$

and let $\mathrm{rsb_i}(\pi)$ (resp. $\mathrm{lsb_i}(\pi)$) be the number of blocks B in π to the right (resp. left) of the block containing i such that the opener of B is smaller than i and the closer of B is greater than i. Then define ros, rob, rcs, rcb, lob, los, lcs, lcb, lsb and rsb as the sum of their coordinate statistics, e.g.

$$\mathrm{ros} = \sum_{\mathrm{i}} \mathrm{ros_i}.$$

For any set of nonnegative integers A and a composed statistic $Stat$ on ordered partitions, we define $stat(A)$ as the sum of the coordinate statistics in A, i.e.,

$$\mathrm{stat}(A) := \sum_{i \in A} stat_i.$$

Now, for any mapping f from $\mathcal{OP}_n^k$ to the set of subsets of $[n]$, we define $stat(f)$ by $stat(f)(\pi) := stat(f(\pi))$.

For a permutation σ of $[n]$, the pair (i, j) is an *inversion* if $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$. Let $\mathrm{inv}\,\sigma$ be the number of inversions in σ and

$$\mathrm{cinv}\sigma = \binom{\mathrm{n}}{2} - \mathrm{inv}\,\sigma.$$

By convention, for a partition π, we put $\mathrm{inv}\ \pi = \mathrm{inv}(\mathrm{perm}(\pi))$ and $\mathrm{cinv}\,\pi = \mathrm{cinv}(\mathrm{perm}(\pi))$. Note that $\mathcal{P}^k = \{\pi \in \mathcal{OP}^k \mid \mathrm{inv}\ \pi = 0\}$ and $\mathrm{bInv}\,\pi = 0$ for each $\pi \in \mathcal{P}^k$.

Given an ordered partition π, let π^r be the ordered partition obtained from π by reversing the order of the blocks. This turns a left (resp. right) opener into a right (resp. left) opener, and likewise for the closers. Moreover, let π^c be the ordered partition obtained by complementing each of the letters in π, that is , by replacing the letter i by $n+1-i$. Then, it is easy to see that $\mathrm{rob}(\pi^{\mathrm{c}}) = \mathrm{rcs}(\pi)$ and $\mathrm{ros}(\pi^{\mathrm{c}}) = \mathrm{rcb}(\pi)$, and likewise for the left and closer statistics. Thus the eight statistics obtained by independently varying left/right, opener/closer and smaller/bigger fall into only two categories when it comes to their distribution on ordered partitions. One of these categories consists of $\mathrm{rob}, \mathrm{lob}, \mathrm{rcs}$ and lcs, and the other contains $\mathrm{ros}, \mathrm{los}, \mathrm{rcb}$ and lcb. Note that these results are completely false on the unordered set partitions.

For instance, we give the values of the coordinate statistics computed on the partition $\pi = 6\,8 - 5 - 1\,4\,7 - 3\,9 - 2$:

$$\begin{array}{rccccccccc}
\pi = & 6\,8 & - & 5 & - & 1\,4\,7 & - & 3\,9 & - & 2 \\
\\
\mathrm{los_i}: & 0\,0 & - & 0 & - & 0\,0\,2 & - & 1\,3 & - & 1 \\
\mathrm{ros_i}: & 4\,4 & - & 3 & - & 0\,2\,2 & - & 1\,1 & - & 0 \\
\mathrm{lob_i}: & 0\,0 & - & 1 & - & 2\,2\,0 & - & 2\,0 & - & 3 \\
\mathrm{rob_i}: & 0\,0 & - & 0 & - & 2\,0\,0 & - & 0\,0 & - & 0 \\
\mathrm{lcs_i}: & 0\,0 & - & 0 & - & 0\,0\,1 & - & 0\,3 & - & 0 \\
\mathrm{rcs_i}: & 2\,3 & - & 1 & - & 0\,1\,1 & - & 1\,1 & - & 0 \\
\mathrm{lcb_i}: & 0\,0 & - & 1 & - & 2\,2\,1 & - & 3\,0 & - & 4 \\
\mathrm{rcb_i}: & 2\,1 & - & 2 & - & 2\,1\,1 & - & 0\,0 & - & 0 \\
\mathrm{lsb_i}: & 0\,0 & - & 0 & - & 0\,0\,1 & - & 1\,0 & - & 1 \\
\mathrm{rsb_i}: & 2\,1 & - & 2 & - & 0\,1\,1 & - & 0\,0 & - & 0
\end{array}$$

Note that there are four block inversions: $\{6,8\} > \{5\}$, $\{6,8\} > \{2\}$, $\{5\} > \{2\}$ and $\{3,9\} > \{2\}$, and two block descents at $i = 1$ and 4; thus $\mathrm{bInv}\,\pi = 4$ and $\mathrm{bmaj}\,\pi = 1 + 4 = 5$. Note also that $\mathrm{bExc}\,\pi = 0$. Moreover, $perm(\pi) = 54132$ and thus $\mathrm{inv}(\pi) = 8$ and $\mathrm{cinv}(\pi) = \binom{5}{2} - 8 = 2$.

Inspired by a statistic mak due to Foata & Zeilberger [4] on the permutations, Steingrímsson introduced its analogous on $\mathcal{OP}_n^k$ as follows:

$$\begin{aligned}
\mathrm{mak} &= \mathrm{ros} + \mathrm{lcs}, \\
\mathrm{lmak} &= n(k-1) - [\mathrm{los} + \mathrm{rcs}], \\
\mathrm{mak}' &= \mathrm{lob} + \mathrm{rcb}, \\
\mathrm{lmak}' &= n(k-1) - [\mathrm{lcb} + \mathrm{rob}].
\end{aligned}$$

The following result was first noticed by Ksavrelof and Zeng in [6]. For completeness, we include a more straightforward proof.

Proposition 2.1. *For any $\pi \in \mathcal{OP}_n^k$ we have*

$$\mathrm{mak} = \mathrm{lmak}' \quad \textit{and} \quad \mathrm{mak}' = \mathrm{lmak}.$$

Proof. For $\pi = B_1 - B_2 - \cdots - B_k \in \mathcal{OP}_n^k$ and $i \in [n]$ we have

$$(\mathrm{los_i} + \mathrm{lob_i} + \mathrm{ros_i} + \mathrm{rob_i})\pi = (\mathrm{lcs_i} + \mathrm{lcb_i} + \mathrm{rcs_i} + \mathrm{rcb_i})\pi = \mathrm{k} - 1.$$

Il follows that

$$\mathrm{los} + \mathrm{lob} + \mathrm{ros} + \mathrm{rob} = \mathrm{lcs} + \mathrm{lcb} + \mathrm{rcs} + \mathrm{rcb} = \mathrm{n}(\mathrm{k} - 1).$$

The Proposition is then equivalent to

$$\mathrm{lob} + \mathrm{los} = \mathrm{lcb} + \mathrm{lcs},$$

which is obvious. □

In view of the above proposition the conjectures in [13] are reduced to the following

Conjecture 2.2 (Steingrímsson)**.** *The following statistics are Euler-Mahonian on $\mathcal{OP}$:*

$$\text{mak} + \text{bInv}, \quad \text{lmak} + \text{bInv}, \quad \text{mak} + \text{bMaj}, \quad \text{lmak} + \text{bMaj},$$
$$\text{cinvLSB} := \text{lsb} + \text{cbInv} + \binom{\text{k}}{2} \quad \text{and} \quad \text{cmajLSB} := \text{lsb} + \text{cbMaj} + \binom{\text{k}}{2},$$

where cbInv $= \binom{\text{k}}{2} - \text{bInv}$ *and* cbMaj $= \binom{k}{2} - \text{bMaj}$. *In other words, the generating functions of the above statistics over $\mathcal{OP}_n^k$ are equal to $[k]_q! S_q(n,k)$.*

Let π be a partition of $[n]$. A *singleton* is the element of a block which has only one element. Now, consider a block B of a partition π whose cardinal is ≥ 2. An element of B is a *strict opener* (resp. *strict closer*) if it is the least (resp. greatest) element of B, and a *transient* if it is neither the least nor greatest element of B.

The sets of strict openers, strict closers, singletons and transients of π will be denoted by $\mathcal{O}(\pi)$, $\mathcal{C}(\pi)$, $\mathcal{S}(\pi)$ and $\mathcal{T}(\pi)$, respectively. The 4-tuple $\lambda(\pi) = (\mathcal{O}(\pi), \mathcal{T}(\pi), \mathcal{S}(\pi), \mathcal{C}(\pi))$ is called the *type* of π. For instance, for the partition $\pi = 3\,5 - 2\,4\,6 - 1 - 7\,8$, we get

$$\mathcal{O}(\pi) = \{2,3,7\}, \quad \mathcal{C}(\pi) = \{5,6,8\}, \quad \mathcal{S}(\pi) = \{1\} \quad \text{and} \quad \mathcal{T}(\pi) = \{4\}.$$

Clearly we have open $= \mathcal{O} \cup \mathcal{S}$ and clos $= \mathcal{C} \cup \mathcal{S}$ therefore we get

$$\begin{aligned} \text{bInv} &= \text{rcs}(\mathcal{O} \cup \mathcal{S}) \quad \text{and} \quad \text{inv} = \text{ros}(\mathcal{O} \cup \mathcal{S}), \\ \text{bExc} &= \text{lcs}(\mathcal{O} \cup \mathcal{S}) \quad \text{and} \quad \text{cinv} = \text{los}(\mathcal{O} \cup \mathcal{S}). \end{aligned}$$

2.2. **Main results.** Consider the following two generating functions of ordered partitions with $k \geq 0$ blocks:

$$\phi_k(a; x, y, t, u) := \sum_{\pi \in \mathcal{OP}^k} x^{(\text{mak}+\text{bInv})\pi} y^{\text{cinvLSB}\,\pi} t^{\text{inv}\,\pi} u^{\text{cinv}\,\pi} a^{|\pi|}, \tag{2.1}$$
$$\varphi_k(a; z, t, u) := \sum_{\pi \in \mathcal{OP}^k} z^{(\text{lmak}+\text{bInv})\pi} t^{\text{inv}\,\pi} u^{\text{cinv}\,\pi} a^{|\pi|}. \tag{2.2}$$

The following is the main result of this paper.

Theorem 2.3. *We have*

$$\phi_k(a; x, y, t, u) = \frac{a^k\,(xy)^{\binom{k}{2}} [k]_{tx,uy}!}{\prod_{i=1}^k (1 - a[i]_{x,y})}, \tag{2.3}$$
$$\varphi_k(a; z, t, u) = \frac{a^k z^{\binom{k}{2}}\,[k]_{tz,u}!}{\prod_{i=1}^k (1 - a[i]_z)}. \tag{2.4}$$

The proof of this theorem will occupy the whole Section 3. We first derive some results on Euler-Mahonian statistics on ordered partitions.

By definition, the combinatorial interpretations of the following specializations of ϕ_k and φ_k are obvious:

$$\begin{aligned}
\sum_{\pi\in\mathcal{OP}^k} q^{(\text{mak}+\text{bInv})\pi}\, a^{|\pi|} &= \phi_k(a;q,1,1,1),\\
\sum_{\pi\in\mathcal{OP}^k} q^{\text{cinvLSB}\,\pi}\, a^{|\pi|} &= \phi_k(a;1,q,1,1),\\
\sum_{\pi\in\mathcal{OP}^k} q^{(\text{mak}+\text{bInv}-\text{inv}+\text{cinv})\pi}\, a^{|\pi|} &= \phi_k(a;q,1,1/q,q),\\
\sum_{\pi\in\mathcal{OP}^k} q^{(\text{cinvLSB}+\text{inv}-\text{cinv})\,\pi}\, a^{|\pi|} &= \phi_k(a;1,q,q,1/q),\\
\sum_{\pi\in\mathcal{OP}^k} q^{(\text{lmak}+\text{bInv})\pi}\, a^{|\pi|} &= \varphi_k(a;q,1,1),\\
\sum_{\pi\in\mathcal{OP}^k} q^{(\text{lmak}+\text{bInv}-\text{inv}+\text{cinv})\pi}\, a^{|\pi|} &= \varphi_k(a;q,1/q,q).
\end{aligned}$$

Applying Theorem 2.3 we see that the right-hand sides of the above six identities are all equal to

$$\frac{a^k\, q^{\binom{k}{2}}[k]_q!}{\prod_{i=1}^k(1-a[i]_q)} = \sum_{n\geq k}[k]_q! S_q(n,k)\, a^n, \tag{2.5}$$

where the last equality follows directly from (1.1). Thus we have proved

Theorem 2.4. *The following six inversion-like statistics are Euler-Mahonian on $\mathcal{OP}$:*

$$\begin{aligned}
\text{mak}+\text{bInv}, &\qquad \text{mak}+\text{bInv}-(\text{inv}-\text{cinv})\ ,\\
\text{lmak}+\text{bInv}, &\qquad \text{lmak}+\text{bInv}+(\text{inv}-\text{cinv})\ ,\\
\text{cinvLSB}, &\qquad \text{cinvLSB}+(\text{inv}-\text{cinv})\ .
\end{aligned}$$

2.3. **Consequence on partitions.** Since a partition is an ordered partition without inversion, so we can derive the following "hard" combinatorial interpretations for q-Stirling numbers by putting $t=0$ and extracting the coefficient of a^n in Theorem 2.3:

$$S_q(n,k) = \sum_{\pi\in\mathcal{P}_n^k} q^{\text{mak}\,\pi} = \sum_{\pi\in\mathcal{P}_n^k} q^{\text{lmak}\,\pi} = \sum_{\pi\in\mathcal{P}_n^k} q^{\text{lsb}\,\pi+\binom{k}{2}}.$$

The first two interpretations were proved by Ksavrelof and Zeng [6]. The third interpretation was first proved by Stanton (see [14]).

Note that by definition $k(1-k) + \mathrm{cinvLSB} = \mathrm{lsb} - \mathrm{bInv}$, then by noticing that the two statistics inv and bInv vanish on (unordered) partitions, we get that

$$\sum_{\pi\in\mathcal{P}^k} q^{\mathrm{mak}\,\pi} a^{|\pi|} = \phi_k(a; q, 1, 0, 1),$$
$$\sum_{\pi\in\mathcal{P}^k} q^{\binom{k}{2}+\mathrm{lsb}\,\pi} a^{|\pi|} = q^{\binom{k}{2}}\phi_k(a; 1, q, 0, 1),$$
$$\sum_{\pi\in\mathcal{P}^k} q^{\mathrm{lmak}\,\pi} a^{|\pi|} = \varphi_k(a; q, 0, 1).$$

Now, applying Theorem 2.3 we see that the right-hand-sides of the above three identities are equal to $\sum_{n\ge k} S_q(n,k)\, a^n$ in view of (2.5).

3. Proof of Theorem 2.3

3.1. **Ordered partitions and walks in digraphs.** Let $\pi = B_1 - B_2 - \cdots - B_k$ be a partition of $[n]$ and i an integer in $[n]$. The restriction $B_j(\le i) := B_j \cap [i]$ of the block B_j is said to be *opened* if $B \not\subseteq [i]$ and $B\cap[i] \neq \emptyset$, *closed* if $B \subseteq [i]$, and *empty* if $B \cap [i] = \emptyset$. The i-th *trace* of π, $T_i(\pi)$, is defined by

$$T_i(\pi) = B_1(\le i) - B_2(\le i) - \cdots - B_k(\le i),$$

where the empty restrictions are not written. The sequence $(T_i(\pi))_{1\le i\le n}$ is called the *trace* of the partition π. We denote by $\mathrm{op}_i\ \pi$ and $\mathrm{cl}_i\ \pi$ the numbers of opened blocks and closed blocks, respectively, in $T_i(\pi)$ and set $\mathcal{F}_i(\pi) = (\mathrm{cl}_i\ \pi, \mathrm{op}_i\ \pi)$ for $1 \le i \le n$ with $\mathcal{F}_0(\pi) = (0,0)$. The sequence $(\mathcal{F}_i(\pi))_{0\le i\le n}$ is called the *form* of the partition π.

For instance, if $\pi = \{6\} - \{3,5,7\} - \{1,4,10\} - \{9\} - \{2,8\}$, then $T_6(\pi) = \{6\} - \{3,5,\cdots\} - \{1,4,\cdots\} - \{2,\cdots\}$, where each opened block has an ellipsis, and we get $\mathcal{F}_6(\pi) = (1,3)$.

Remark 3.1. *Given the form of a partition, it is easy to deduce its type, and reversely. Indeed, let $i \in [n]$ and suppose that $\mathcal{F}_{i-1}(\pi) = (k,l)$, then*

$$\mathcal{F}_i(\pi) = \begin{cases} (k, l+1), & \text{if } i \in \mathcal{O}(\pi); \\ (k+1, l), & \text{if } i \in \mathcal{S}(\pi); \\ (k, l), & \text{if } i \in \mathcal{T}(\pi); \\ (k+1, l-1), & \text{if } i \in \mathcal{C}(\pi). \end{cases}$$

Note that if $l = 0$, then i can be neither a strict closer nor a transient ($l = 0$ means that all the blocks in $T_{i-1}\,\pi$ are closed).

For any integer $k \ge 0$, let D_k be the digraph with vertex set $V_k = \{(i,j) \in \mathbb{N}^2 \mid i+j \le k\}$, and there is an edge in D_k from (i,j) to (i',j') if and only if $(i',j') = (i,j)$ with $j > 0$ or $(i',j') \in \{(i,j+1),\ (i+1,j),\ (i+1,j-1)\}$. It is obvious that the number of vertices of D_k is equal to

$$\widehat{k} := 1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}.$$

Let $v_1, \cdots, v_{\widehat{k}}$ be the vertices of D_k arranged according to the following order: $(i,j) \le (i',j')$ if and only if $i + j < i' + j'$ or ($i + j = i' + j'$ and $j \ge j'$). For instance, we get

$v_1 = (0,0)$, $v_2 = (0,1)$, $v_3 = (1,0)$, $v_4 = (0,2)$, $v_5 = (1,1)$, $v_6 = (2,0), \cdots, v_{\widehat{k}} = (k,0)$. An illustration of D_k is given in Figure 2.

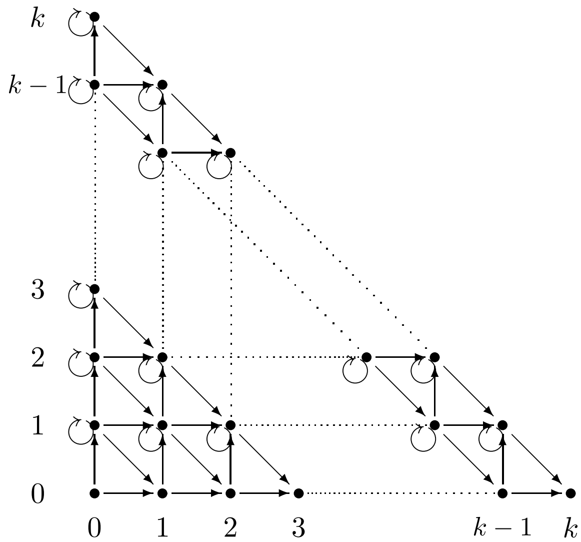

FIGURE 1. The digraph D_k

A path of length n in D_k is a finite sequence $(s_0, s_1, \ldots, s_n)$ of points in V_k such that (s_i, s_{i+1}) is an edge for $i = 0, \ldots, n-1$. The step (s_i, s_{i+1}) is called *North* (resp. *East*, *South-East* and *Null*) if $s_{i+1} = (x_i, y_i + 1)$ (resp. $s_{i+1} = (x_i + 1, y_i)$, $s_{i+1} = (x_i + 1, y_i - 1)$ and $s_{i+1} = s_i$). The number x_i and y_i are respectively the *abscissa* and the *height* of the step (s_i, s_{i+1}).

Definition 3.2. *A path in D_k from $v_1 = (0,0)$ to $v_{\widehat{k}} = (k,0)$ is called a path of depth k. Let Ω_n^k be the set of paths of depth k and length n and $\Omega^k = \bigcup_{n \geq 0} \Omega_n^k$ be the set of all paths of depth k.*

The following result just follows from the definition of paths.

Proposition 3.3. *For $n \geq k \geq 0$, the forms of partitions in $\mathcal{OP}_n^k$ are exactly the paths in Ω_n^k.*

We can visualize a path by drawing a segment from s_{i-1} to s_i in the x-y plan. For instance, the path

$$\omega = ((0,0),\ (0,1),\ (0,2),\ (0,3),\ (0,3),\ (0,3),\ (1,3),\ (2,2),\ (3,1),\ (4,1),\ (5,0))$$

is illustrated in Figure 2.

By definition, an ordered partition on $[i]$ with two kinds of blocks, opened or closed, is called a *trace* on $[i]$. Let T_{i-1} be a trace on $[i-1]$ and suppose that $\mathcal{F}(T_{i-1}) = (k,l)$, with $k, l \geq 0$. There are several possibilities to insert the element i in T_{i-1} according to the nature of i. The element i could be :

(a) a strict closer (resp. a transient): there are $p_i = l$ possibilities to close with i (resp. insert i in) one of the l opened blocks of T_{i-1}.

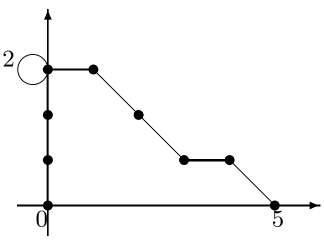

FIGURE 2. A path in Ω_{10}^5, where 2 means two successive Null steps at $(0,3)$.

(b) a singleton (resp. a strict opener): there are $p_i = k + l + 1$ possibilities to insert the singleton $\{i\}$ (resp. open a block with i) in T_{i-1} (before all the blocks, between two blocks, or after all the blocks).

We observe that if $l = 0$, then in case (a), there are none possibility to add i ($p_i = 0$), which is natural because all the blocks in T_{i-1} are closed.

Remark 3.4. *The number of ways p_i to add the element i, according to its "nature", in a trace T_{i-1} on $[i-1]$, depends only on $\mathcal{F}(T_{i-1})$. If $\mathcal{F}(T_{i-1}) = (k,l)$, then $p_i = l$ (resp. $p_i = k + l + 1$) if we want insert i as a transient or a strict closer (resp. a strict opener or a singleton).*

We assume the possibilities to add an element (according its nature) in a trace are arranged from left to right. Namely, if we insert a singleton or open a block (resp. insert a transient or close a block), the spaces (resp. opened blocks) which correspond to the possibilities are arranged from left to right.

For instance, $T_6 = \{6\} - \{3,5,\cdots\} - \{1,4,\cdots\} - \{2,\cdots\}$ is a trace on $[6]$, then $\mathcal{F}(T_6)$=(1,3) and there are:

(i) 5 possibilities to open a block or insert a singleton in T_6.

$$\begin{array}{lccccccccc} T_5 = & & \{6\} & & \{3,5,\cdots\} & & \{1,4,\cdots\} & & \{2,\cdots\} & \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{choice} & 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

(ii) 3 possibilities to close a block or add a transient in T_6. Namely,

$$\begin{array}{lcccc} T_6 = & \{6\} & \{3,5\cdots\} & \{1,4,\cdots\} & \{2,\cdots\} \\ & & \uparrow & \uparrow & \uparrow \\ \text{choice} & & 1 & 2 & 3 \end{array}$$

Definition 3.5. *A path diagram of depth k and length n is a pair (ω,ξ), where ω is a path of depth k and length n and $\xi = (\xi_i)_{1\le i\le n}$ is a sequence of integers such that $1 \le \xi_i \le q$ if the i-th step of ω is Null or South-East of height q, and $1 \le \xi_i \le p + q + 1$ if the i-th step of ω is North or East of abscissa p and height q.*

Denote by Δ_n^k the set of path diagrams of depth k and length n and by $\Delta^k = \bigcup_{n\ge 0} \Delta_n^k$ the set of path diagrams of depth k.

The mapping ψ: Let $n \ge k \ge 1$. Given a path diagram $h = (\omega,\xi) \in \Delta_n^k$, we associate a partition $\psi(h) \in \mathcal{OP}_n^k$ by constructing successively its traces T_i for $1 \le i \le n$ as follows:

(1) Set $T_0 = \emptyset$.
(2) For $1 \le i \le n$, we construct T_i from T_{i-1} by the following process. Suppose that $s_{i-1} = (p, q)$ and the i-th step of ω is :
 (i) North (resp. East), then we open a block with i (resp. insert the singleton $\{i\}$) in T_{i-1} according to the choice ξ_i.
 (ii) South-East (resp. Null), then we close with i (resp. insert i as a transient in) an opened block of T_{i-1} according to the choice ξ_i.
(3) Set $\psi(h) = T_n$

For instance, if $h = (\omega, \xi)$ where ω is the path of Figure 1 and $\xi = (1, 2, 1, 2, 1, 1, 1, 2, 4, 1)$, then the step by step construction of $\psi(h)$ goes as follows:

i	step$_i$	ξ_i	T_i
1	*North*	1	$\{1, \cdots\}$
2	*North*	2	$\{1, \cdots\} - \{2, \cdots\}$
3	*North*	1	$\{3, \cdots\} - \{1, \cdots\} - \{2, \cdots\}$
4	*Null*	2	$\{3, \cdots\} - \{1, 4, \cdots\} - \{2, \cdots\}$
5	*Null*	1	$\{3, 5, \cdots\} - \{1, 4, \cdots\} - \{2, \cdots\}$
6	*East*	1	$\{6\} - \{3, 5, \cdots\} - \{1, 4, \cdots\} - \{2, \cdots\}$
7	*South-East*	1	$\{6\} - \{3, 5, 7\} - \{1, 4, \cdots\} - \{2, \cdots\}$
8	*South-East*	2	$\{6\} - \{3, 5, 7\} - \{1, 4, \cdots\} - \{2, 8\}$
9	*East*	4	$\{6\} - \{3, 5, 7\} - \{1, 4, \cdots\} - \{9\} - \{2, 8\}$
10	*South-East*	1	$\{6\} - \{3, 5, 7\} - \{1, 4, 10\} - \{9\} - \{2, 8\}$.

Thus $\psi(h) = \{6\} - \{3, 5, 7\} - \{1, 4, 10\} - \{9\} - \{2, 8\}$.

Theorem 3.6. *For each $n \ge k \ge 1$, the mapping $\psi : h = (\omega, \xi) \mapsto \pi$ is a bijection from Δ_n^k to $\mathcal{OP}_n^k$ such that:*
if the i-th step of ω is North or East of abscissa p and height q, then $i \in (\mathcal{O} \cup \mathcal{S})(\pi)$ and

$$(\mathrm{lcs} + \mathrm{rcs})_{\mathrm{i}}(\pi) = \mathrm{p}, \quad (\mathrm{lsb} + \mathrm{rsb})_{\mathrm{i}}(\pi) = \mathrm{q}, \quad \mathrm{los}_{\mathrm{i}}(\pi) = \xi_{\mathrm{i}} - 1 \quad \textit{and} \quad \mathrm{ros}_{\mathrm{i}}(\pi) = \mathrm{p} + \mathrm{q} + 1 - \xi_{\mathrm{i}};$$

if the i-th step of ω is South-East or Null of abscissa p and height q, then $i \in (\mathcal{T} \cup \mathcal{C})(\pi)$ and

$$(\mathrm{lcs} + \mathrm{rcs})_{\mathrm{i}}(\pi) = \mathrm{p}, \quad (\mathrm{lsb} + \mathrm{rsb})_{\mathrm{i}}(\pi) = \mathrm{q} - 1, \quad \mathrm{lsb}_{\mathrm{i}}(\pi) = \xi_{\mathrm{i}} - 1 \quad \textit{and} \quad \mathrm{rsb}_{\mathrm{i}}(\pi) = \mathrm{q} - \xi_{\mathrm{i}}.$$

Proof. By Remark 3.4 and Proposition 3.3, it is easy to see that the above algorithm is well defined. Suppose we are constructing the ordered partition π and we arrive at the i-th step of the construction. The i-th step of w is a step with initial vertex (p, q). At this step of the construction, there are exactly $p + q$ blocks in T_{i-1}, whose p (resp. q) are closed (resp. opened), and all the elements in T_{i-1} are strictly inferior than i. Suppose that the i-th step of w is a step of type:

(i) North or East : then $1 \leq a_i \leq p+q+1$ and by remark 3.1, the element i will be a strict opener or a singleton of the partition π. It's clear that $(\mathrm{lcs}+\mathrm{rcs})_\mathrm{i}(\pi)$ (resp. $(\mathrm{lsb}+\mathrm{rsb})_\mathrm{i}(\pi)$) is equal to the number of closed (resp. opened) blocks in $T_{i-1}(\pi)$. Thus, $(\mathrm{lcs}+\mathrm{rcs})_\mathrm{i}(\pi) = \mathrm{p}$ and $(\mathrm{lsb}+\mathrm{rsb})_\mathrm{i}(\pi) = \mathrm{q}$.
Because all the blocks in T_{i-1} have opener smaller than i, then $\mathrm{los}_\mathrm{i}\,\pi$ (resp. $\mathrm{ros}_\mathrm{i}\,\pi$) is just the number of blocks in $T_i(\pi)$ on the left (resp. right) of the block which contains i. Thus, because we open a block with the element i or add the singleton $\{i\}$ between the (ξ_i-1)-th and (ξ_i)-th blocks of T_{i-1}, we get $(\mathrm{los}_\mathrm{i}, \mathrm{ros}_\mathrm{i})(\pi) = (\xi_\mathrm{i}-1, \mathrm{p}+\mathrm{q}+1-\xi_\mathrm{i})$.

(ii) South-East or Null : then $1 \leq a_i \leq q$, and by remark 3.1, the element i will be a transient or a strict closer of the partition π. By the same arguments as case (i), we get $(\mathrm{lcs}+\mathrm{rcs})_\mathrm{i}(\pi) = \mathrm{p}$. Remark $(\mathrm{lsb}+\mathrm{rsb})_\mathrm{i}(\pi)$ is equal to the number of opened blocks in $T_i(\pi)$ which don't contain the element i, thus because we insert i in one of the opened block in T_{i-1}, we get $(\mathrm{lsb}+\mathrm{rsb})_\mathrm{i}(\pi) = \mathrm{q}-1$. Moreover, $\mathrm{lsb}_\mathrm{i}(\pi)$ (resp. $\mathrm{rsb}_\mathrm{i}(\pi)$) is equal to the number of opened blocks in $T_i(\pi)$ on the left (resp. on the right) of the block which contains i. Because we insert i in the ξ_i-th opened block in $T_{i-1}(\pi)$, we get $(\mathrm{lsb}_\mathrm{i}, \mathrm{rsb}_\mathrm{i})(\pi) = (\xi_\mathrm{i}-1, \mathrm{q}-\xi_\mathrm{i})$.

□

3.2. **Generating functions of walks.** For $0 \leq k \leq n$, let $\mathbf{t} = (t_1, t_2, t_3, t_4, t_5, t_6, t_7)$ and

$$Q_{n,k}(\mathbf{t}) := \sum_{\pi \in \mathcal{OP}_n^k} t_1^{(\mathrm{lcs}+\mathrm{rcs})(\mathcal{O}\cup\mathcal{S})\pi}\, t_2^{(\mathrm{lcs}+\mathrm{rcs})(\mathcal{T}\cup\mathcal{C})\pi}\, t_3^{\mathrm{rsb}(\mathcal{T}\cup\mathcal{C})\pi} \times t_4^{\mathrm{lsb}(\mathcal{T}\cup\mathcal{C})\pi}\, t_5^{\mathrm{ros}(\mathcal{O}\cup\mathcal{S})\pi}\, t_6^{\mathrm{los}(\mathcal{O}\cup\mathcal{S})\pi}\, t_7^{(\mathrm{lsb}+\mathrm{rsb})(\mathcal{O}\cup\mathcal{S})\pi}. \tag{3.1}$$

Given a path ω, define the weight $\mathrm{v}(\omega)$ of ω to be the product of the weights of all its steps, where the weight of a step of abscissa i and height j is:

$$v(\omega) = \begin{cases} t_1^i\, t_7^j\, [i+j+1]_{t_5,t_6} & \text{if the step is North or East;} \\ t_2^i\, [j]_{t_3,t_4} & \text{if the step is Null or South-East.} \end{cases} \tag{3.2}$$

It follows easily from Theorem 3.6 that

$$\sum_{\omega \in \Omega_n^k} \mathrm{v}(\omega) = Q_{n,k}(\mathbf{t}) \ .$$

Denote by $|\omega|$ the length of the path ω. Then, using the above identity, we get

$$Q_k(a;\mathbf{t}) := \sum_{n\geq 0} Q_{n,k}(\mathbf{t})\, a^n = \sum_{w \in \Omega^k} \mathrm{v}(w)\, a^{|\omega|} \ . \tag{3.3}$$

The *adjacency matrix* A_k of D_k relative to the valuation v is the $\widehat{k} \times \widehat{k}$ matrix defined by

$$A_k(i,j) = \begin{cases} \mathrm{v}(v_i, v_j) & \text{if } (v_i, v_j) \text{ is an edge of } D_k; \\ 0 & \text{otherwise.} \end{cases}$$

Applying transfer-matrix method (see e.g. [12, Theorem 4.7.2]), we derive

$$Q_k(a;\mathbf{t}) = \frac{(-1)^{1+\widehat{k}} \det(I - aA_k; \widehat{k}, 1)}{\det(I - aA_k)}, \tag{3.4}$$

where $(B; i, j)$ denotes the matrix obtained by removing the i-th row and j-th column of B and I is the $\widehat{k} \times \widehat{k}$ identity matrix.

In order to prove Theorem 2.3 we need to evaluate (3.4) in the following special cases:

$$f_k(a; x, y, t, u) = Q_k(a; x, x, x, y, t, u, y), \tag{3.5}$$

$$g_k(a; z, t, u) = Q_k(a; 1, z, 1, z, t, u, 1). \tag{3.6}$$

Let A'_k and A''_k be the adjacency matrix of D_k relative to the weight function v$'$ and v$''$ obtained from the weight function v by making the substitution (3.5) and (3.6), respectively. Namely, the weights v$'(e)$ and v$''(e)$ of an edge $e = ((i, j), (i', j'))$ of D_k with initial vertex (i, j) are :

$$\mathrm{v}'(e) = \begin{cases} x^i\, y^j\, [i+j+1]_{t,u} & \text{if } (i', j') = (i, j+1) \text{ or } (i+1, j)\ ; \\ x^i\, [j]_{x,y} & \text{if } (i', j') = (i, j) \text{ or } (i+1, j-1), \end{cases}$$

and

$$\mathrm{v}''(e) = \begin{cases} [i+j+1]_{t,u} & \text{if } (i', j') = (i, j+1) \text{ or } (i+1, j)\ ; \\ z^i [j]_z & \text{if } (i', j') = (i, j) \text{ or } (i+1, j-1). \end{cases}$$

Now, for each $k \geq 0$ let

$$M_k = I - aA'_k \quad \text{and} \quad N_k = I - aA''_k.$$

Then by (3.4), (3.5) and (3.6) we have

$$f_k(a; x, y, t, u) = \frac{(-1)^{1+\widehat{k}} \det(M_k; \widehat{k}, 1)}{\det M_k}, \tag{3.7}$$

$$g_k(a; z, t, u) = \frac{(-1)^{1+\widehat{k}} \det(N_k; \widehat{k}, 1)}{\det N_k} \tag{3.8}$$

for each $k \geq 1$.

Since M_n and N_n are upper triangular matrices it is easy to see that for each $n \geq 1$

$$\det M_n = \prod_{m=1}^{n} \prod_{i=0}^{m} (1 - ax^i [m-i]_{x,y}), \tag{3.9}$$

$$\det N_n = \prod_{m=1}^{n} \prod_{k=0}^{n-m} (1 - az^k [m]_q). \tag{3.10}$$

The evaluation of $\det(M_n; \widehat{n}, 1)$ and $\det(N_n; \widehat{n}, 1)$ is not simple(see last section).

Theorem 3.7. *Let $n \geq 1$ be a positive integer. Then*

$$\det(M_n; \widehat{n}, 1) = (-1)^{\binom{n}{2}} a^n x^{\binom{n}{2}} [n]_{t,u}! \prod_{m=1}^{n-1} \prod_{i=1}^{m} (1 - ax^i [m-i+1]_{x,y}), \tag{3.11}$$

Theorem 3.8. *Let $n \geq 1$ be a positive integer. Then*

$$\det(N_n; \widehat{n}, 1) = (-1)^{\binom{n}{2}} a^n\, [n]_{t,u}! \prod_{m=1}^{n-1} \prod_{k=1}^{n-m} (1 - az^{k-1} [m]_z). \tag{3.12}$$

It is now trivial to obtain the following result.

Corollary 3.9. *For $k \geq 0$, we have*

$$f_k(a;x,y,t,u) = \frac{a^k x^{\binom{k}{2}} [k]_{t,u}!}{\prod_{i=1}^{k}(1-a[i]_{x,y})}, \tag{3.13}$$

$$g_k(a;z,t,u) = \frac{a^k\, [k]_{t,u}!}{\prod_{i=1}^{k}(1-az^{k-i}[i]_z)}\,. \tag{3.14}$$

To see the relation between f_k and ϕ_k, g_k and φ_k we need to establish the following

Lemma 3.10. *The following functional identities hold on $\mathcal{OP}_n^k$:*

$$\begin{aligned} \mathrm{mak} + \mathrm{bInv} &= (\mathrm{lcs} + \mathrm{rcs}) + \mathrm{rsb}(\mathcal{T} \cup \mathcal{C}) + \mathrm{inv}, \\ \mathrm{lmak} + \mathrm{bInv} &= n(k-1) - (\mathrm{lcs} + \mathrm{rcs})(\mathcal{T} \cup \mathcal{C}) - \mathrm{lsb}(\mathcal{T} \cup \mathcal{C}) - \mathrm{cinv}, \\ \mathrm{cinvLSB} &= (\mathrm{lsb} + \mathrm{rsb})(\mathcal{O} \cup \mathcal{S}) + \mathrm{lsb}(\mathcal{T} \cup \mathcal{C}) + \mathrm{inv} + 2\,\mathrm{cinv}. \end{aligned}$$

Proof. By definition we have

$$\begin{aligned} \mathrm{mak} + \mathrm{bInv} &= \mathrm{lcs} + \mathrm{ros} + \mathrm{rcs}(\mathcal{O} \cup \mathcal{S}) \\ &= (\mathrm{lcs} + \mathrm{rcs}) + \mathrm{ros} + (\mathrm{rcs}(\mathcal{O} \cup \mathcal{S}) - \mathrm{rcs}) \\ &= (\mathrm{lcs} + \mathrm{rcs}) + \mathrm{ros} - \mathrm{rcs}(\mathcal{T} \cup \mathcal{C}) \\ &= (\mathrm{lcs} + \mathrm{rcs}) + (\mathrm{ros} - \mathrm{rcs})(\mathcal{T} \cup \mathcal{C}) + \mathrm{ros}(\mathcal{O} \cup \mathcal{S}) \\ &= (\mathrm{lcs} + \mathrm{rcs}) + \mathrm{rsb}(\mathcal{T} \cup \mathcal{C}) + \mathrm{ros}(\mathcal{O} \cup \mathcal{S}). \end{aligned}$$

Also

$$\begin{aligned} n(k-1) - (\mathrm{lmak} + \mathrm{bInv}) &= (\mathrm{los} + \mathrm{rcs}) - \mathrm{rcs}(\mathcal{O} \cup \mathcal{S}) = \mathrm{los} + \mathrm{rcs}(\mathcal{T} \cup \mathcal{C}) \\ &= \mathrm{los}(\mathcal{O} \cup \mathcal{S}) + \mathrm{rcs}(\mathcal{T} \cup \mathcal{C}) + \mathrm{los}(\mathcal{T} \cup \mathcal{C}) \\ &= \mathrm{cinv} + (\mathrm{lcs} + \mathrm{rcs})(\mathcal{T} \cup \mathcal{C}) + (\mathrm{los} - \mathrm{lcs})(\mathcal{T} \cup \mathcal{C}) \\ &= \mathrm{cinv} + (\mathrm{lcs} + \mathrm{rcs})(\mathcal{T} \cup \mathcal{C}) + \mathrm{lsb}(\mathcal{T} \cup \mathcal{C}), \end{aligned}$$

and

$$\begin{aligned} \mathrm{cinvLSB} &= k(k-1) + \mathrm{lsb} - \mathrm{bInv} \\ &= 2(\mathrm{inv} + \mathrm{cinv}) + \mathrm{lsb} - \mathrm{rcs}(\mathcal{O} \cup \mathcal{S}) \\ &= 2(\mathrm{inv} + \mathrm{cinv}) + (\mathrm{lsb} + \mathrm{rsb})(\mathcal{O} \cup \mathcal{S}) \\ &\qquad - (\mathrm{rcs} + \mathrm{rsb})(\mathcal{O} \cup \mathcal{S}) + \mathrm{lsb}(\mathcal{T} \cup \mathcal{C}) \\ &= 2(\mathrm{inv} + \mathrm{cinv}) + (\mathrm{lsb} + \mathrm{rsb})(\mathcal{O} \cup \mathcal{S}) - \mathrm{ros}(\mathcal{O} \cup \mathcal{S}) + \mathrm{lsb}(\mathcal{T} \cup \mathcal{C}) \\ &= (\mathrm{lsb} + \mathrm{rsb})(\mathcal{O} \cup \mathcal{S}) + \mathrm{lsb}(\mathcal{T} \cup \mathcal{C}) + \mathrm{inv} + 2\,\mathrm{cinv}. \end{aligned}$$

The proof is thus completed. $\square$

Now, we derive from (3.5) and (3.6) that

$$\begin{aligned} f_k(a;x,y,t,u) &= \sum_{\pi \in \mathcal{OP}^k} x^{(\mathrm{lcs}+\mathrm{rcs}+\mathrm{rsb}(\mathcal{T}\cup\mathcal{C}))\,\pi} y^{((\mathrm{lsb}+\mathrm{rsb})(\mathcal{O}\cup\mathcal{S})+\mathrm{lsb}(\mathcal{T}\cup\mathcal{C}))\,\pi} t^{\mathrm{inv}\,\pi} u^{\mathrm{cinv}\,\pi} a^{|\pi|}, \\ g_k(a;z,t,u) &= \sum_{\pi \in \mathcal{OP}^k} z^{(\mathrm{lcs}+\mathrm{rcs}+\mathrm{lsb})(\mathcal{T}\cup\mathcal{C})\pi} t^{\mathrm{inv}\,\pi} u^{\mathrm{cinv}\,\pi} a^{|\pi|}. \end{aligned}$$

It follows from the above lemma the following

Lemma 3.11. *The following identities hold:*

$$\phi_k(a;x,y,t,u) = f_k(a;x,y,xyt,uy^2), \tag{3.15}$$

$$\varphi_k(a;z,t,u) = g_k(az^{k-1};1/z,t,u/z). \tag{3.16}$$

Finally Theorem 2.3 follows immediately from Corollary 3.9 and Lemma 3.11. Therefore in order to prove Theorem 2.3 it remains to prove Theorem 3.7 and Theorem 3.8.

3.3. **Proof of Theorem 3.7.** The matrix M_n can be defined recursively by

$$M_0 = (1)\,, \quad M_n = \left(\begin{array}{c|c} M_{n-1} & \overline{M}_{n-1} \\ \hline O_{n+1,\widehat{n-1}} & \widehat{M}_{n-1} \end{array}\right), \tag{3.17}$$

where $n \geq 1$,

$$\widehat{M}_{n-1} = \big(\delta_{ij} - ax^{i-1}[n+1-i]_{x,y}(\delta_{ij}+\delta_{i+1,j})\big)_{1\leq i,j\leq n+1} \tag{3.18}$$

and $\overline{M}_{n-1}$ is the $\widehat{n-1} \times (n+1)$ matrix

$$\overline{M}_{n-1} = \left(\begin{array}{c} O_{\widehat{n-2},n+1} \\ \hline \check{M}_{n-1} \end{array}\right)$$

with the $n \times (n+1)$ matrix

$$\check{M}_{n-1} = \big(-ax^{i-1}y^{n-i}[n]_{t,u}(\delta_{ij}+\delta_{i+1,j})\big)_{1\leq i\leq n,\,1\leq j\leq n+1}\,. \tag{3.19}$$

Here δ_{ij} stands for the Kronecker delta and $O_{m,n}$ denotes the $m \times n$ zero matrix. For instance, we get

$$M_1 = \left(\begin{array}{c|cc} 1 & -a & -a \\ \hline 0 & 1-a & -a \\ 0 & 0 & 1 \end{array}\right)$$

and

$$M_2 = \left(\begin{array}{ccc|ccc} 1 & -a & -a & 0 & 0 & 0 \\ 0 & 1-a & -a & -ay(t+u) & -ay(t+u) & 0 \\ 0 & 0 & 1 & 0 & -ax(t+u) & -ax(t+u) \\ \hline 0 & 0 & 0 & 1-a(x+y) & -a(x+y) & 0 \\ 0 & 0 & 0 & 0 & 1-ax & -ax \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Let

$$\mathrm{K}_n = \widehat{n} - 1 = \frac{n(n+3)}{2},$$

and let $P_n = (M_n; \widehat{n}, 1)$, i.e the $\mathrm{K}_n \times \mathrm{K}_n$ matrix obtained from M_n by deleting the $\widehat{n}$th row and the first column. P_n can be defined as follows.

$$P_n = \left(\begin{array}{c|c} P_{n-1} & \overline{P}_{n-1} \\ \hline X_{n-1} & \widehat{P}_{n-1} \end{array} \right)$$

Here $\overline{P}_{n-1}$ is a $\mathrm{K}_{n-1} \times (n+1)$ matrix, X_{n-1} is a $(n+1) \times \mathrm{K}_{n-1}$ matrix, and $\widehat{P}_{n-1}$ is a $(n+1) \times (n+1)$ matrix. We shall compute $\det P_n$ by the following well-known formula for any block matrix with an invertible square matrix A,

$$\det \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \det A \cdot \det \left(D - CA^{-1}B \right).$$

Since the entries of $CA^{-1}B$ are also written by minors, we guess these entries and prove it by induction (see Theorem 3.12). Before we proceed to the proof, we give some examples.

$$P_1 = \begin{pmatrix} -a & -a \\ 1-a & -a \end{pmatrix}$$

and

$$P_2 = \left(\begin{array}{cc|ccc} -a & -a & 0 & 0 & 0 \\ 1-a & -a & -ay(t+u) & -ay(t+u) & 0 \\ \hline 0 & 1 & 0 & -ax(t+u) & -ax(t+u) \\ 0 & 0 & 1-a(x+y) & -a(x+y) & 0 \\ 0 & 0 & 0 & 1-ax & -ax \end{array} \right).$$

Thus, looking at P_2 as the block matrix composed of P_1, X_1, $\overline{P}_1$ and $\widehat{P}_1$, we have

$$\overline{P}_1 = \begin{pmatrix} 0 & 0 & 0 \\ -ay(t+u) & -ay(t+u) & 0 \end{pmatrix}$$

and

$$\widehat{P}_1 = \begin{pmatrix} 0 & -ax(t+u) & -ax(t+u) \\ 1-a(x+y) & -a(x+y) & 0 \\ 0 & 1-ax & -ax \end{pmatrix}.$$

Since $\overline{P}_n$ is an $\mathrm{K}_n \times (n+2)$ matrix, we can write

$$\overline{P}_n = \left(\frac{O_{\mathrm{K}_{n-1}, n+2}}{U_n} \right),$$

where U_n is the $(n+1) \times (n+2)$ matrix composed of the last $(n+1)$ rows of $\overline{P}_n$. For $1 \le k \le n+2$, let

$$P_n^k = \left(\begin{array}{c|c} P_{n-1} & \overline{P}_{n-1} \\ \hline X_{n-1} & \widehat{P}_{n-1}^k \end{array} \right)$$

denote the $\mathrm{K}_n \times \mathrm{K}_n$ matrix obtained from P_n by replacing the right-most column with the kth column of $\overline{P}_n$. Here $\widehat{P}^k_{n-1}$ is the $(n+1)\times(n+1)$ matrix obtained from $\widehat{P}_{n-1}$ by replacing the right-most column with the kth column of U_n. For example,

$$P_2^2 = \left(\begin{array}{cc|ccc} -a & -a & 0 & 0 & 0 \\ 1-a & -a & -ay(t+u) & -ay(t+u) & 0 \\ \hline 0 & 1 & 0 & -ax(t+u) & 0 \\ 0 & 0 & 1-a(x+y) & -a(x+y) & -ay^2(t^2+tu+t^2) \\ 0 & 0 & 0 & 1-ax & -axy(t^2+tu+t^2) \end{array}\right).$$

Here our key result is as follows:

Theorem 3.12. *Let $n \geq 1$ be a positive integer. Then we have*

$$\frac{\det P_n}{\det P_{n-1}} = (-1)^{n-1} a x^{n-1} [n]_{t,u} \prod_{i=1}^{n-1} (1 - ax^i [n-i]_{x,y}), \tag{3.20}$$

and

$$\frac{\det P_n^k}{\det P_n} = a\, x^{\frac{(k-1)(k-2)}{2} - \frac{n(n-1)}{2}}\, y^{\frac{(n+1-k)(n+2-k)}{2}}\, [n+1]_{t,u} \begin{bmatrix} n+1 \\ k-1 \end{bmatrix}_{x,y} \tag{3.21}$$

for $1 \leq k \leq n$,

$$\frac{\det P_n^{n+1}}{\det P_n} = a\, y\, [n+1]_{t,u}\, [n]_{x,y} \tag{3.22}$$

and $\det P_n^{n+2} = 0$.

We need the following:

Lemma 3.13. *For $0 \leq m \leq n$,*

$$\sum_{k=0}^{m} (-1)^{m-k} x^{\binom{k}{2}} y^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{x,y} \prod_{i=0}^{k-1} \left\{1 - ax^i [n-i]_{x,y}\right\} \prod_{i=k}^{m-1} \left\{-ax^i [n-i]_{x,y}\right\}$$
$$= x^{\binom{m}{2}} y^{\binom{n-m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_{x,y} \prod_{i=1}^{m} \left\{1 - ax^i [n-i]_{x,y}\right\}. \tag{3.23}$$

Proof. Note that

$$[n]_{x,y} = y^{n-1} [n]_{x/y}, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_{x,y} = y^{kn-k^2} \begin{bmatrix} n \\ k \end{bmatrix}_{x/y}.$$

Since $\binom{n-k}{2} + \binom{k}{2} + kn - k^2 = \binom{n}{2}$, setting $c = ay^{n-1}$ and $q = x/y$, we can rewrite (3.23) as follows:

$$\sum_{k=0}^{m} (-1)^{m-k} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1} \left\{1 - cq^i [n-i]_q\right\} \prod_{i=k}^{m-1} \left\{-cq^i [n-i]_q\right\}$$
$$= q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \prod_{i=1}^{m} \left\{1 - cq^i [n-i]_q\right\}. \tag{3.24}$$

Setting

$$X = 1 + \frac{cq^n}{1-q}, \quad Y = \frac{c}{1-q+cq^n}, \quad Z = \frac{cq^n}{1-q},$$

then $1 - cq^i[n-i]_q = X(1 - Yq^i)$ and $-cq^i[n-i]_q = Z(1 - q^{i-n})$. Hence, in (3.24) making the following substitutions:

$$\prod_{i=0}^{k-1} \left\{1 - cq^i[n-i]_q\right\} = X^k (Y;q)_k,$$
$$\prod_{i=1}^{m} \left\{1 - cq^i[n-i]_q\right\} = X^m (Yq;q)_m,$$
$$\prod_{i=k}^{m-1} \left\{-cq^i[n-i]_q\right\} = (-1)^m Z^{m-k} q^{\binom{m}{2}-mn} \frac{(q^{n-m+1};q)_m}{(q^{-n};q)_k},$$

and writing the q-binomial coefficients as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = (-1)^k q^{kn-k(k-1)/2} \frac{(q^{-n};q)_k}{(q;q)_k}, \quad \begin{bmatrix} n \\ m \end{bmatrix}_q = (-1)^m q^{mn-m(m-1)/2} \frac{(q^{-n};q)_m}{(q;q)_m},$$

we see, after simplifying, that identity (3.24) is equivalent to the special case $Y = c/(1 - q + cq^n)$ of the identity:

$$\sum_{k=0}^{m} \frac{(Y;q)_k}{(q;q)_k} Y^{m-k} = \frac{(Yq;q)_m}{(q;q)_m},$$

which can be easily verified by induction. $\square$

Proof of Theorem 3.12. We proceed by induction on n. When $n = 1$, by a direct computation we obtain $\det P_1 = a$, $\det P_1^1 = \det P_1^2 = a^2\, y\, [2]_{t,u}$ and $\det P_1^3 = 0$. This shows the theorem is true when $n = 1$. Let n be an integer ≥ 2. Assume the theorem is true for $n-1$.

(i) We get

$$\det P_n = \det \left(\begin{array}{c|c} P_{n-1} & \overline{P}_{n-1} \\ \hline X_{n-1} & \widehat{P}_{n-1} \end{array} \right) = \det P_{n-1} \cdot \det \left(\widehat{P}_{n-1} - X_{n-1} P_{n-1}^{-1} \overline{P}_{n-1} \right)$$

and

$$\det P_n^k = \det \left(\begin{array}{c|c} P_{n-1} & \overline{P}_{n-1} \\ \hline X_{n-1} & \widehat{P}_{n-1}^k \end{array} \right) = \det P_{n-1} \cdot \det \left(\widehat{P}_{n-1}^k - X_{n-1} P_{n-1}^{-1} \overline{P}_{n-1} \right).$$

(ii) By direct computation we can see that the (i,j)th entry of $X_{n-1} P_{n-1}^{-1} \overline{P}_{n-1}$ $(1 \leq i, j \leq n+1)$ is equal to

$$\begin{cases} \frac{\det P_{n-1}^j}{\det P_{n-1}} & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By the induction hypothesis, the $(1,j)$th entry of $X_{n-1}P_{n-1}^{-1}\overline{P}_{n-1}$ equals

$$\begin{cases} a\,x^{\binom{j-1}{2}-\binom{n-1}{2}}\,y^{\binom{n+1-j}{2}}\,[n]_{t,u}\begin{bmatrix} n \\ j-1\end{bmatrix}_{x,y} & \text{if } 1\le j\le n-1, \\ a\,y\,[n-1]_{x,y}\,[n]_{t,u} & \text{if } j=n, \\ 0 & \text{if } j=n+1. \end{cases} \tag{3.25}$$

(iii) Put $W_{n-1}^{k}=\widehat{P}_{n-1}^{k}-X_{n-1}P_{n-1}^{-1}\overline{P}_{n-1}$ and $W_{n-1}=\widehat{P}_{n-1}-X_{n-1}P_{n-1}^{-1}\overline{P}_{n-1}$. Then, by (i), we have $\frac{\det P_n^k}{\det P_{n-1}}=\det W_{n-1}^k$ and $\frac{\det P_n}{\det P_{n-1}}=\det W_{n-1}$. By (3.19) and (3.25), we can see that the $(1,j)$th entry of W_{n-1}^k is

$$-a\,x^{\binom{j-1}{2}-\binom{n-1}{2}}\,y^{\binom{n+1-j}{2}}[n]_{t,u}\begin{bmatrix} n \\ j-1\end{bmatrix}_{x,y}$$

for $1\le j\le n$, and the $(1,n+1)$th entry is 0 (the top row does not depend on k). It is also easy to see that the $(1,j)$th entry of W_{n-1} is

$$-a\,x^{\binom{j-1}{2}-\binom{n-1}{2}}\,y^{\binom{n+1-j}{2}}[n]_{t,u}\begin{bmatrix} n \\ j-1\end{bmatrix}_{x,y}$$

for $1\le j\le n+1$.

(iv) We claim that

$$\det W_{n-1}=(-1)^{n-1}ax^{n-1}[n]_{t,u}\prod_{i=1}^{n-1}(1-ax^{i}[n-i]_{x,y}).$$

In fact, the (i,j)th entry of W_{n-1} is

$$\begin{cases} -a\,y^{\frac{(n-j)(n-j+1)}{2}}\,x^{\frac{(j-1)(j-2)}{2}-\frac{(n-1)(n-2)}{2}}[n]_{t,u}\begin{bmatrix} n \\ j-1\end{bmatrix}_{x,y} & \text{if } i=1 \text{ and } 1\le j\le n+1, \\ 1-ax^{j-1}[n+1-j]_{x,y} & \text{if } i=j+1 \text{ and } 1\le j\le n, \\ -ax^{j-2}[n+2-j]_{x,y} & \text{if } i=j \text{ and } 2\le j\le n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if we expand $\det W_{n-1}$ along the top row, then we obtain

$$\begin{aligned}\det W_{n-1}=-ax^{-\frac{(n-1)(n-2)}{2}}&[n]_{t,u}\\ &\times\sum_{j=1}^{n+1}(-1)^{j+1}y^{\frac{(n-j)(n-j+1)}{2}}x^{\frac{(j-1)(j-2)}{2}}\begin{bmatrix} n \\ j-1\end{bmatrix}_{x,y}\det W_{n-1}(1;j).\end{aligned}$$

If we use

$$\det W_{n-1}(1;j)=\prod_{\nu=0}^{j-2}(1-ax^{\nu}[n-\nu]_{x,y})\prod_{\nu=j-1}^{n-1}(-ax^{\nu}[n-\nu]_{x,y}),$$

then we obtain

$$\det W_{n-1} = -ax^{-\frac{(n-1)(n-2)}{2}}[n]_{t,u}\sum_{j=1}^{n+1}(-1)^{j+1}y^{\frac{(n-j)(n-j+1)}{2}}x^{\frac{(j-1)(j-2)}{2}}\begin{bmatrix} n \\ j-1 \end{bmatrix}_{x,y}$$

$$\times\prod_{\nu=0}^{j-2}(1-ax^{\nu}[n-\nu]_{x,y})\prod_{\nu=j-1}^{n-1}(-ax^{\nu}[n-\nu]_{x,y})$$

$$= (-1)^{n-1}ax^{n-1}[n]_{t,u}\prod_{i=1}^{n-1}(1-ax^{i}[n-i]_{x,y})$$

by (3.23). Thus, by (i), we conclude that

$$\frac{\det P_n}{\det P_{n-1}} = \det W_{n-1} = (-1)^{n-1}ax^{n-1}[n]_{t,u}\prod_{i=1}^{n-1}(1-ax^{i}[n-i]_{x,y}). \tag{3.26}$$

(v) We claim that

$$\frac{\det P_n^k}{\det P_n} = \frac{\det W_{n-1}^k}{\det W_{n-1}} = a\,x^{\frac{(k-1)(k-2)}{2}-\frac{n(n-1)}{2}}\,y^{\frac{(n+1-k)(n+2-k)}{2}}\,[n+1]_{t,u}\begin{bmatrix} n+1 \\ k-1 \end{bmatrix}_{x,y}$$

for $1 \le k \le n$. Because the rightmost column of $\widehat{P}_{n-1}^k$ is the kth column of U_n, we have the $(i, n+1)$th entry of $\widehat{P}_{n-1}^k$ is

$$\begin{cases} -ay^n[n+1]_{t,u} & \text{if } i=2, \\ 0 & \text{otherwise,} \end{cases}$$

when $k=1$,

$$\begin{cases} -ay^{n+2-k}x^{k-2}[n+1]_{t,u} & \text{if } i=k, \\ -ay^{n+1-k}x^{k-1}[n+1]_{t,u} & \text{if } i=k+1, \\ 0 & \text{otherwise,} \end{cases}$$

when $2 \le k \le n$,

$$\begin{cases} -ayx^{n-1}[n+1]_{t,u} & \text{if } i=n+1, \\ 0 & \text{otherwise,} \end{cases}$$

when $k = n+1$, and all zero when $k = n+2$. By the induction hypothesis, the $(1, n+1)$th entry of $X_{n-1}P_{n-1}^{-1}\overline{P}_{n-1}$ is $\frac{\det P_{n-1}^{n+1}}{\det P_{n-1}} = 0$. Thus the $(n+1)$th column of $W_{n-1}^k = \widehat{P}_{n-1}^k - X_{n-1}P_{n-1}^{-1}\overline{P}_{n-1}$ equals the $(n+1)$th column of $\widehat{P}_{n-1}^k$.

(a) When $k=1$, we expand $\det W_{n-1}^1$ along the $(n+1)$th column, then, by direct computation, we obtain

$$\det W_{n-1}^1 = (-1)^{n+3}(-ay^n[n+1]_{t,u})\det W_{n-1}(2;n+1)$$

By expanding $\det W_{n-1}^1(2;n+1)$ along the top row we obtain

$$\det W_{n-1}(2;n+1) = \left(-a\,y^{\frac{n(n-1)}{2}}\,x^{-\frac{(n-1)(n-2)}{2}}[n]_{t,u}\right)\prod_{\nu=1}^{n-1}(1-ax^{\nu}[n-\nu]_{x,y}).$$

Thus we conclude that

$$\det W_{n-1}^1 = (-1)^{n-1}a^2 y^{\frac{n(n+1)}{2}} x^{-\frac{(n-1)(n-2)}{2}} [n]_{t,u} \times [n+1]_{t,u} \prod_{\nu=1}^{n-1}(1 - ax^\nu [n-\nu]_{x,y}). \tag{3.27}$$

By (3.26), this implies

$$\frac{\det W_{n-1}^1}{\det W_{n-1}} = ay^{\frac{n(n+1)}{2}} x^{-\frac{n(n-1)}{2}} [n+1]_{t,u}$$

which is the desired identity.

(b) When $2 \le k \le n$, we expand $\det W_{n-1}^k$ along the $(n+1)$th column, then we obtain

$$\begin{aligned}\det W_{n-1}^k &= (-1)^{k+n+1}(-ay^{n+2-k}x^{k-2}[n+1]_{t,u}) \det W_{n-1}(k; n+1) \\ &+ (-1)^{k+n+2}(-ay^{n+1-k}x^{k-1}[n+1]_{t,u}) \det W_{n-1}(k+1; n+1).\end{aligned} \tag{3.28}$$

By expanding along the top row, we obtain

$$\begin{aligned}\det W_{n-1}(k; n+1) &= -ax^{-\frac{(n-1)(n-2)}{2}} [n]_{t,u} \sum_{j=1}^{k-1} (-1)^{j+1} y^{\frac{(n-j)(n-j+1)}{2}} x^{\frac{(j-1)(j-2)}{2}} \\ &\times \begin{bmatrix} n \\ j-1 \end{bmatrix}_{x,y} \det W_{n-1}(1, k; j, n+1),\end{aligned}$$

$$\begin{aligned}\det W_{n-1}(k+1; n+1) &= -ax^{-\frac{(n-1)(n-2)}{2}} [n]_{t,u} \sum_{j=1}^{k} (-1)^{j+1} y^{\frac{(n-j)(n-j+1)}{2}} x^{\frac{(j-1)(j-2)}{2}} \\ &\times \begin{bmatrix} n \\ j-1 \end{bmatrix}_{x,y} \det W_{n-1}(1, k+1; j, n+1),\end{aligned}$$

where $W_{n-1}(1,k;j,n+1) = W_{n-1}(k;n+1)(1;j)$ and $W_{n-1}(1,k+1;j,n+1) = W_{n-1}(k+1;n+1)(1;j)$. If we use

$$\begin{aligned}\det W_{n-1}(1, k; j, n+1) &= \prod_{\nu=0}^{j-2}(1 - ax^\nu[n-\nu]_{x,y}) \prod_{\nu=j-1}^{k-3} (-ax^\nu[n-\nu]_{x,y}) \\ &\times \prod_{\nu=k-1}^{n-1} (1 - ax^\nu[n-\nu]_{x,y}),\end{aligned}$$

$$\begin{aligned}\det W_{n-1}(1, k+1; j, n+1) &= \prod_{\nu=0}^{j-2}(1 - ax^\nu[n-\nu]_{x,y}) \prod_{\nu=j-1}^{k-2} (-ax^\nu[n-\nu]_{x,y}) \\ &\times \prod_{\nu=k}^{n-1}(1 - ax^\nu[n-\nu]_{x,y}),\end{aligned}$$

then, by (3.23), we obtain

$$\det W_{n-1}(k;n+1) = (-1)^{k-1} a x^{\frac{(k-2)(k-3)}{2} - \frac{(n-1)(n-2)}{2}} y^{\frac{(n-k+2)(n-k+1)}{2}} \times [n]_{t,u} \begin{bmatrix} n \\ k-2 \end{bmatrix}_{x,y} \prod_{\nu=1}^{n-1} (1 - ax^{\nu}[n-\nu]_{x,y}),$$

$$\det W_{n-1}(k+1;n+1) = (-1)^{k} a x^{\frac{(k-1)(k-2)}{2} - \frac{(n-1)(n-2)}{2}} y^{\frac{(n-k)(n-k+1)}{2}} \times [n]_{t,u} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{x,y} \prod_{\nu=1}^{n-1} (1 - ax^{\nu}[n-\nu]_{x,y}).$$

Thus, from (3.28), we conclude that

$$\begin{aligned} \det W_{n-1}^{k} =& (-1)^{n-1} a^2 x^{-\frac{(n-1)(n-2)}{2} + \frac{(k-1)(k-2)}{2}} y^{\frac{(n+1-k)(n+2-k)}{2}} [n]_{t,u}[n+1]_{t,u} \\ & \times \left(y^{n-k+2} \begin{bmatrix} n \\ k-2 \end{bmatrix}_{x,y} + x^{k-1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{x,y} \right) \prod_{\nu=1}^{n-1} (1 - ax^{\nu}[n-\nu]_{x,y}) \\ &= (-1)^{n-1} a^2 x^{-\frac{(n-1)(n-2)}{2} + \frac{(k-1)(k-2)}{2}} y^{\frac{(n+1-k)(n+2-k)}{2}} \\ & \quad \times [n]_{t,u}[n+1]_{t,u} \begin{bmatrix} n+1 \\ k-1 \end{bmatrix}_{x,y} \prod_{\nu=1}^{n-1} (1 - ax^{\nu}[n-\nu]_{x,y}). \end{aligned}$$

Using (3.26), we obtain

$$\frac{\det W_{n-1}^{k}}{\det W_{n-1}} = a x^{-\frac{n(n-1)}{2} + \frac{(k-1)(k-2)}{2}} y^{\frac{(n+1-k)(n+2-k)}{2}} [n+1]_{t,u} \begin{bmatrix} n+1 \\ k-1 \end{bmatrix}_{x,y},$$

which is the desired identity.

(c) When $k = n+1$, we also expand $\det W_{n-1}^{k}$ along the $(n+1)$th column and repeat the same argument. It is not hard to obtain

$$\frac{\det W_{n-1}^{n+1}}{\det W_{n-1}} = ay[n+1]_{t,u}[n]_{x,y}.$$

The details are left to the reader.

(d) When $k = n+2$, $\det \widehat{P}_{n-1}^{k}$ vanishes since all the entries of the last column of $\det \widehat{P}_{n-1}^{k}$ are zero.

This proves the theorem is true for n. By induction we conclude that the theorem is true for all $n \geq 1$. This completes the proof. □

Since $\det(P_1) = a$, Theorem 3.7 (3.11) follows easily from (3.20). □

3.4. **Proof of Theorem 3.8.** Let $F = \{F_n\}_{n=1}^{\infty}$ be a sequence of non-zero functions in finitely many variables $v_1, v_2, \ldots$. We use the convention that $F_n! = \prod_{k=1}^{n} F_k$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \begin{cases} \frac{F_n!}{F_k! F_{n-k}!}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

We prove Theorem 3.8 (3.12) by considering the following matrix $N_n(x,a)$, which generalize the matrix N_n (set $x=1$ and $F_n=[n]_{t,u}$ to obtain N_n). Let $N_n(x,a)$ be the matrix defined inductively as follows:

$$N_0(x,a)=(x)$$

and

$$N_n(x,a)=\left(\begin{array}{c|c} N_{n-1}(x,a) & \overline{N}_{n-1}(x,a) \\ \hline O_{n+1,\widehat{n-1}} & \widehat{N}_{n-1}(x,a) \end{array}\right) \tag{3.29}$$

where $\widehat{N}_{n-1}(x,a)$ is the $(n+1)\times(n+1)$ matrix defined by

$$\widehat{N}_{n-1}(x,a)=\left(x\delta_{ij}-aq^{i-1}[n+1-i]_q(\delta_{ij}+\delta_{i+1,j})\right)_{1\le i,j\le n+1} \tag{3.30}$$

and $\overline{N}_{n-1}(x,a)$ is the $\widehat{n-1}\times(n+1)$ matrix

$$\left(\begin{array}{c} O_{\widehat{n-2},n+1} \\ \hline \check{N}_{n-1} \end{array}\right)$$

with the $n\times(n+1)$ matrix

$$\check{N}_{n-1}=\left(-aF_n\cdot(\delta_{ij}+\delta_{i+1,j})\right)_{1\le i\le n,\,1\le j\le n+1}. \tag{3.31}$$

For instance, we get

$$N_2(x,a)=\begin{pmatrix} x & -aF_1 & -aF_1 & 0 & 0 & 0 \\ 0 & x-a & -a & -aF_2 & -aF_2 & 0 \\ 0 & 0 & x & 0 & -aF_2 & -aF_2 \\ 0 & 0 & 0 & x-a(1+q) & -a(1+q) & 0 \\ 0 & 0 & 0 & 0 & x-aq & -aq \\ 0 & 0 & 0 & 0 & 0 & x \end{pmatrix}.$$

Let $\dot{N}_n(x,a)$ denote the matrix obtained from $N_n(x,a)$ by deleting the $\widehat{n}$th row and the first column. Then the following theorem is sufficient to prove our result. Here our strategy is as follows. We regard $\det N_n(x,a)$ as a polynomial in x and find all linear factors. Finally we check the leading coefficient in the both sides.

Theorem 3.14. *We have*

$$\det\dot{N}_n(x,a)=(-1)^{\frac{n(n-1)}{2}}a^n\,F_n!\,x^n\prod_{m=1}^{n-1}\prod_{k=1}^{n-m}\left(x-aq^{k-1}[m]_q\right). \tag{3.32}$$

Then by setting $x=1$ and $F_n=[n]_{t,u}$ we obtain Theorem 3.8 (3.12).

For instance, we have

$$\det \dot{N}_1(x,a) = \det \begin{pmatrix} -aF_1 & -aF_1 \\ \\ x-a & -a \end{pmatrix} = a\,F_1\,x$$

and

$$\det \dot{N}_2(x,a) = \det \begin{pmatrix} -aF_1 & -aF_n & 0 & 0 & 0 \\ \\ x-a & -a & -aF_2 & -aF_2 & 0 \\ \\ 0 & x & 0 & -aF_2 & -aF_2 \\ \\ 0 & 0 & x-a(1+q) & -a(1+q) & 0 \\ \\ 0 & 0 & 0 & x-aq & -aq \end{pmatrix}$$
$$= -a^2\,F_1F_2\,x^2(x-a).$$

Fix positive integers m and k. Define the row vectors $X_n^{m,k}(x,a)$ of degree $\widehat{n}$ as follows: For $1 \le i \le n+1$ and $1 \le j \le i$, the $\left(\frac{i(i-1)}{2}+j\right)$th entry of $X_n^{m,k}(x,a)$ is equal to

$$X_{i,j}^{m,k} = (-1)^{i+m+k} a q^{-(m+k-1)(i-m-k)+\binom{j-k}{2}} \frac{F_{i-m-k}!}{[i-m-k]_q!} \begin{bmatrix} i-1 \\ m+k-1 \end{bmatrix}_F \begin{bmatrix} m \\ j-k \end{bmatrix}_q. \quad (3.33)$$

Here we use the convention that $F_n! = [n]_q! = 1$ if $n \le 0$. For example, if $n=3$, $m=k=1$, then

$$X_3^{1,1}(x,a) = \left(0,1,1,-\frac{F_2}{q},-\frac{F_2}{q},0,\frac{F_2F_3}{q^2[2]_q!},\frac{F_2F_3}{q^2[2]_q!},0,0\right).$$

Lemma 3.15. *Let n be a positive integer. Let m and k be positive integers such that $1 \le m \le n-1$ and $1 \le k \le n-m$. Then we have*

$$X_n^{m,k}(x,a)\,N_n(x,a) = (x - aq^{k-1}[m]_q)\,X_n^{m,k}(x,a). \quad (3.34)$$

Before we proceed to the proof of the lemma, we see it in an example. If $n=2$ and $m=k=1$, then we have

$$\left(0,1,1,-\frac{F_2}{q},-\frac{F_2}{q},0\right)\begin{pmatrix} x & -F_1 & -F_1 & 0 & 0 & 0 \\ 0 & x-a & -a & -aF_2 & -aF_2 & 0 \\ 0 & 0 & x & 0 & -aF_2 & -aF_2 \\ 0 & 0 & 0 & x-a(1+q) & -a(1+q) & 0 \\ 0 & 0 & 0 & 0 & x-aq & -aq \\ 0 & 0 & 0 & 0 & 0 & x \end{pmatrix}$$
$$=\left(0,x-a,x-a,-\frac{F_2}{q}(x-a),-\frac{F_2}{q}(x-a),0\right).$$

Proof of Lemma 3.15. We proceed by induction on n. When $n=0$ or $n=1$, our claim is easy to check by direct computation. Assume (3.34) is true upto $n-1$. Then the first $\widehat{n-1}$ entries of $X_n^{m,k}(x,a)N_n(x,a)$ agree with those of $(x-q^{k-1}[m]_q)X_n^{m,k}(x,a)$ by the induction hypothesis. So we have to check the last $n+1$ entries. In fact we verify the following three cases.

(i) If $i=n+1$ and $j=1$, then the $\left(\frac{n(n+1)}{2}+1\right)$th entry of $X_n^{m,k}(x,a)N_n(x,a)$ is equal to

$$(-a\,F_n)X_{n,1}^{m,k}+(x-a[n]_q)X_{n+1,1}^{m,k}.$$

Note that the coefficient $\left[\begin{smallmatrix} m \\ 1-k \end{smallmatrix}\right]_q$ becomes zero unless $k=1$. Thus, by direct computation, one can easily check that this sum equals

$$(-1)^{n+m+k+1}aq^{-(m+k-1)(n-m-k+1)+\binom{1-k}{2}}$$
$$\times\frac{F_{n+1-m-k}!}{[n+1-m-k]_q!}\begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F\begin{bmatrix} m \\ 1-k \end{bmatrix}_q(x-a[m+k-1]_q).$$

(ii) If $i=n+1$ and $2\le j\le n$, then the $\left(\frac{n(n+1)}{2}+j\right)$th entry of $X_n^{m,k}(x,a)N_n(x,a)$ is equal to

$$(-aF_n)X_{n,j-1}^{m,k}+(-aF_n)X_{n,j}^{m,k}$$
$$+\left(-aq^{j-2}[n-j+2]_q\right)X_{n+1,j-1}^{m,k}+\left(x-aq^{j-1}[n-j+1]_q\right)X_{n+1,j}^{m,k}.$$

By direct computation, one can easily check this equals

$$(-1)^{n+m+k+1}aq^{-(m+k-1)(n-m-k+1)+\binom{j-k}{2}}$$
$$\times\frac{F_{n+1-m-k}!}{[n+1-m-k]_q!}\begin{bmatrix} n \\ m+k-1 \end{bmatrix}_F\begin{bmatrix} m \\ j-k \end{bmatrix}_q\left(x-aq^{k-1}[m]_q\right).$$

(iii) If $i=j=n+1$, then the $\frac{(n+1)(n+2)}{2}$th entry of $X_n^{m,k}(x,a)N_n(x,a)$ is equal to

$$(-aF_n)X_{n,n}^{m,k}+\left(-aq^{n-1}\right)X_{n+1,n}^{m,k}+xX_{n+1,n+1}^{m,k}.$$

One can easily check this is always equal to zero.

Thus this completes the proof of our lemma. $\square$

Corollary 3.16. *Let n be a positive integer. Then there exists a polynomial $\varphi(x)$ such that*

$$\det \dot{N}_n(x,a) = \varphi(x) \prod_{m=1}^{n-1} \prod_{k=1}^{n-m} \left(x - aq^{k-1}[m]_q\right).$$

Proof. Let $\dot{X}_n^{m,k}(x,a)$ (resp. $\ddot{X}_n^{m,k}(x,a)$) denote the vector of degree $\widehat{n}-1$ obtained from $X_n^{m,k}(x,a)$ by deleting the last (resp. first) entry. Then, by (3.34), we obtain

$$\dot{X}_n^{m,k}(x,a)\dot{N}_n(x,a) = \left(x - aq^{k-1}[m]_q\right)\, \ddot{X}_n^{m,k}(x,a).$$

By substituting $x = q^{k-1}[m]$ into this identity we obtain

$$\dot{X}_n^{m,k}(aq^{k-1}[m]_q,a)\dot{N}_n(aq^{k-1}[m]_q,a) = \vec{0}.$$

Since $\dot{X}_n^{m,k}(x,a)$ is a non-zero vector when $1 \le m \le n-1$ and $1 \le k \le n-m$, $\dot{N}_n(aq^{p-1}[m]_q,a)$ is singular, which means $\det \dot{N}_n(aq^{k-1}[m]_q,a) = 0$. Thus we conclude that $\det \dot{N}_n(x,a)$ is divisible by $x - aq^{k-1}[m]_q$, which immediately implies our corollary. $\square$

Proposition 3.17. *Let n be a positive integer. Then there exists a polynomial $\psi(x)$ such that*

$$\det \dot{N}_n(x,a) = \psi(x)\, x^n \prod_{m=1}^{n-1} \prod_{k=1}^{n-m} \left(x - aq^{k-1}[m]_q\right).$$

Proof. By Corollary 3.16, we only need to show that $\det \dot{N}_n(x,a)$ is divisible by x^n. We show this by the following column transformations on $\dot{N}_n(x,a)$. First note that $\dot{N}_n(x,a)$ has $\widehat{n}-1$ columns. For each $i = 1,\dots,n$, let Col_i denote the set of columns $j = \frac{i(i+1)}{2}, \frac{i(i+1)}{2}+1, \dots, \frac{(i+1)(i+2)}{2}-1$. We perform the following column transformations in each block Col_i. For each i, we subtract the $\frac{i(i+1)}{2}$th column from the $\left(\frac{i(i+1)}{2}+1\right)$th column, then subtract the $\left(\frac{i(i+1)}{2}+1\right)$th column from the $\left(\frac{i(i+1)}{2}+2\right)$th column, and so on, until we subtract the $\left(\frac{(i+1)(i+2)}{2}-2\right)$th column from the $\left(\frac{(i+1)(i+2)}{2}-1\right)$th column. Then each entry of the $\left(\frac{(i+1)(i+2)}{2}-1\right)$th column becomes $\pm x$. For example, if we perform this operation to $\dot{N}_3(x,a)$ which looks like

$$\left(\begin{array}{cc|ccc|cccc}
-aF_1 & -aF_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x-a & -a & -aF_2 & -aF_2 & 0 & 0 & 0 & 0 & 0 \\
0 & x & 0 & -aF_2 & -aF_2 & 0 & 0 & 0 & 0 \\
0 & 0 & x-a[2]_q & -a[2]_q & 0 & -aF_3 & -aF_3 & 0 & 0 \\
0 & 0 & 0 & x-aq & -aq & 0 & -aF_3 & -aF_3 & 0 \\
0 & 0 & 0 & 0 & x & 0 & 0 & -aF_3 & -aF_3 \\
0 & 0 & 0 & 0 & 0 & x-a[3]_q & -a[3]_q & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x-aq[2]_q & -aq[2]_q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x-aq^2 & -aq^2
\end{array}\right),$$

then we obtain

$$\left(\begin{array}{cc|ccc|cccc}
-aF_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x-a & -x & -aF_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & 0 & -aF_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x-a[2]_q & -x & x & -aF_3 & 0 & 0 & 0 \\
0 & 0 & 0 & x-aq & -x & 0 & -aF_3 & 0 & 0 \\
0 & 0 & 0 & 0 & x & 0 & 0 & -aF_3 & 0 \\
0 & 0 & 0 & 0 & 0 & x-a[3]_q & -x & x & -x \\
0 & 0 & 0 & 0 & 0 & 0 & x-aq[2]_q & -x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x-aq^2 & -x
\end{array}\right).$$

This is always true. One can easily check the last column of each block becomes $\pm x$ after these elementary transformations using the definition of $\widehat{N}_{n-1}(x,a)$ and $\overline{N}_{n-1}(x,a)$ in (3.30) and (3.31). Thus, by taking the determinant of $\dot{N}_n(x,a)$, we can factor out x from each block Col_i, $i=1,2,\dots,n$, and we conclude that $\det \dot{N}_n(x,a)$ is divisible by x^n. $\square$

Now we are in position to complete the proof of Theorem 3.14.

Proof of Theorem 3.14. To complete the proof of Theorem 3.14, we need to show that the degree of $\det \dot{N}_n(x,a)$ is $\frac{n(n+1)}{2}$ as a polynomial in x, and the leading coefficient of $\det \dot{N}_n(x,a)$ is equal to $(-1)^{\frac{n(n-1)}{2}} a^n F_n!$. Let $\mathrm{K}_n = \widehat{n}-1$ which is the degree of the matrix $\dot{N}_n(x,a)$. Let $\dot{b}_{ij}$ denote the (i,j)th entry of $\dot{N}_n(x,a)$. By the definition of determinants we have

$$\det \dot{N}_n(x,a) = \sum_{\pi\in S_{\mathrm{K}_n}} \operatorname{sgn}\pi\, \dot{b}_{\pi(1)1}\dot{b}_{\pi(2)2}\cdots \dot{b}_{\pi(\mathrm{K}_n)\mathrm{K}_n}.$$

We use the two-line notation

$$\pi = \begin{pmatrix} 1 & 2 & \dots & \mathrm{K}_n \\ \pi(1) & \pi(2) & \dots & \pi(\mathrm{K}_n) \end{pmatrix}$$

to express a permutation π of letters $[\mathrm{K}_n]$. For each j, if $\pi(j)=j+1$, then the entry $\dot{b}_{\pi(j)j}$ is of degree 1 as a polynomial in x, and otherwise it is a constant. Thus $\det \dot{N}_n(x,a)$ is apparently of at most $\mathrm{K}_n - 1 = \frac{(n+1)(n+2)}{2} - 2$ degree as a polynomial in x. For example $\dot{N}_3(x,a)$ looks as follows.

$$\left(\begin{array}{cc|ccc|cccc}
-aF_1 & \boldsymbol{-aF_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\boldsymbol{x-a} & -a & -aF_2 & -aF_2 & 0 & 0 & 0 & 0 & 0 \\
0 & x & 0 & -aF_2 & \boldsymbol{-aF_2} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & \boldsymbol{x-a[2]_q} & -a[2]_q & 0 & -aF_3 & -aF_3 & 0 & 0 \\
0 & 0 & 0 & \boldsymbol{x-aq} & -aq & 0 & -aF_3 & -aF_3 & 0 \\
0 & 0 & 0 & 0 & x & 0 & 0 & -aF_3 & \boldsymbol{-aF_3} \\
\hline
0 & 0 & 0 & 0 & 0 & \boldsymbol{x-a[3]_q} & -a[3]_q & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{x-aq[2]_q} & -aq[2]_q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{x-aq^2} & -aq^2
\end{array}\right),$$

Our first claim is that $\det \dot{N}_n(x,a)$ is a polynomial of degree $\frac{n(n+1)}{2}$. Let Col_i, $i=1,2,\dots,n$, be as in the previous proof. Note that Col_i includes $i+1$ columns. We

claim that $\pi(j) = j+1$ can happen at most i column indices j in each block Col_i. Otherwise $\dot b_{\pi(1)1}\dot b_{\pi(2)2}\cdots\dot b_{\pi(\mathrm{K}_n)\mathrm{K}_n}$ vanishes. In fact, assume that $\pi(j) = j+1$ for all j in a certain block Col_i. Then this must be the case for the block Col_{i+1}. There is no other choice if we assume $\dot b_{\pi(1)1}\dot b_{\pi(2)2}\cdots\dot b_{\pi(\mathrm{K}_n)\mathrm{K}_n}$ is nonzero. And this must be also the case for the block Col_{i+2}, and so on. Finally we have to take $\pi(j) = j+1$ for all j in the block Col_n, but this is impossible. Thus we reach a contradiction. We conclude that the degree of $\det \dot N_n(x,a)$ is at most $\frac{n(n+1)}{2}$. In fact there is a permutation which realize this degree, i.e.

$$\pi = \left(\begin{array}{cc|ccc|cc|ccccc} 1 & 2 & 3 & 4 & 5 & \dots & \dots & \mathrm{K}_{n-1}+1 & \mathrm{K}_{n-1}+2 & \dots & \mathrm{K}_n-1 & \mathrm{K}_n \\ 2 & 1 & 4 & 5 & 3 & \dots & \dots & \mathrm{K}_{n-1}+2 & \mathrm{K}_{n-1}+3 & \dots & \mathrm{K}_n & \mathrm{K}_{n-1}+1 \end{array}\right).$$

It is easy to see that this π is the only permutation with which $\dot b_{\pi(1)1}\dot b_{\pi(2)2}\cdots\dot b_{\pi(\mathrm{K}_n)\mathrm{K}_n}$ does not vanish and of degree $\frac{n(n+1)}{2}$. Thus we conclude that the leading coefficient of $\det \dot N_n(x,a)$ equals

$$\operatorname{sgn}\pi\cdot(-1)^n\, a^n\, F_n!.$$

This immediately implies the resulting identity (3.32). □

Remark 3.18. *One may notice that M_n in* (3.17) *and N_n in* (3.29) *are in a similar form, but our methods to evaluate them are far from parallel. It seems that the first method does not work with the matrix N_n since we can't guess the entries of $CA^{-1}N$ as we did in* (3.21)*. Meanwhile, the second method does not work with the matrix M_n at this point since even if we generalize M_n to $M_n(x,a)$, we don't know the general form of the eigenvectors of $M_n(x,a)$. The reader can find the general guidance about matrix evaluation in* [5]*. We may say that the second proof follows this general philosophy.*

References

[1] Carlitz (L.), *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc., **76** (1954), 332–350.

[2] Carlitz (L.), *A combinatorial property of q-Eulerian numbers*, Amer. Math. Monthly, **82** (1975), 51–54. 332–350.

[3] Clarke (R.), Steingrímsson (E.) and Zeng (J.), *New Euler-Mahonian statistics on permutations and words*, Adv. in Appl. Math. **18** (1997), no. 3, 237–270.

[4] Foata (D.) and Zeilberger (D.), *Denert's permutation statistic is indeed Euler-Mahonian*, Studies in Appl. Math., **83** (1990), 31-59.

[5] Krattenthaler (C.), "Advanced determinant calculus", *Sem. Lothar. Combin.* **42** ("The Andrews Festschrift") (1999), Article B42q.

[6] Ksavrelof (G.) and Zeng (J.), *Nouvelles statistiques de partitions pour les q-nombres de Stirling de seconde espèce*, Discrete Math., Vol. 256, Issue 3, 2002, 743–758.

[7] Milne (S.), *Restricted growth functions, rank row matching of partition lattices, and q-Stirling numbers*, Adv. Math., **43** (1982), 173-196.

[8] Remmel (B.) and Wachs (M.) , *Rook theory, generalized Stirling numbers and (p,q)-analogues*, Electron. J. Combin., **11** (2004), no.1, Research paper 84.

[9] Sagan (B.), *A maj statistics for set partitions*, European J. Combin., **12** (1991), 69-79.

[10] Simion (S.) and Stanton (D.), *Specializations of generalized Laguerre polynomials*, SIAM J. MATH. ANAL., **25** (1994), 712-719.

[11] Simion (S.) and Stanton (D.), *Octabasic Laguerre polynomials and permutation statistics*, J. of Computational and Applied Math. **68** (1996), 297-329.

[12] Stanley (R. P.), Enumerative combinatorics I, Cambridge Studies in Advanced Mathematics 49, 1997.

[13] Steingrímsson (E.), *Statistics on ordered partitions of sets*, preprint, 1999, available at Arxiv:math.CO/0605670.
[14] Wachs (M.) and White (D.), *p, q-Stirling numbers and set partition statistics*, J. Combin. Theory Ser. A, **56** (1991), 27-46.
[15] Wachs (M.), *σ-Restricted Growth Functions and p,q-stirling numbers*, J. Combin. Theory Ser. A, **68** (1994), 470-480.
[16] White (D.), *Interpolating Set Partition Statistics*, J. Combin. Theory Ser. A, **68** (1994), 262-295.
[17] Zeng (J.) and Zhang (C. G.), *A q-analog of Newton's series, Stirling functions and Eulerian functions*, Results in Math., 25 (1994), 370-391.
[18] Zeng (J.), *Euler-Mahonian statistics on ordered partitions and q-Meixner polynomials*, Talk given at the 52th Séminaire Lotharingien de Combinatoire, March 2004.

Faculty of Education, Tottori University, Koyama, Tottori, Japan
E-mail address: ishikawa@fed.tottori-u.ac.jp

Institut Camille Jordan, Université Claude Bernard (Lyon I), F-69622, Villeurbanne Cedex, France
E-mail address: anisse@math.univ-lyon1.fr

Institut Camille Jordan, Université Claude Bernard (Lyon I), F-69622, Villeurbanne Cedex, France
E-mail address: zeng@math.univ-lyon1.fr